Hilbert modules over locally C^*-algebras

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Abstract

In the present paper the notion of a Hilbert module over a locally C^*-algebra is discussed and some results are obtained on this matter. In particular, we give a detailed proof of the known result that the set of adjointable endomorphisms of such modules is itself a locally C^*-algebra.

Introduction

General theory of Hilbert modules over an arbitrary C^*-algebra was constructed in the papers of W. Paschke [20] and M. Rieffel [24] as a natural generalization of the Hilbert spaces theory. This generalization arises if a C^*-algebra takes place of the field of scalars $\mathbf{C}$. Theory of Hilbert modules is applied to many fields of mathematics, in particular, to theory of C^*-algebras [25, 21], to theory of vector bundles [3], to theory of index of elliptic operators [18, 26], to K-theory [19, 9], to KK-theory of G. Kasparov [10], to theory of quantum groups and unbounded operators [28], to some physical problems [16, 14] etc. There is also a number of papers dedicated to theory of Hilbert C^*-modules proper (see, for example, [17, 6, 7, 13]).

Other topological $*$-algebras, for example, locally C^*-algebras and some group algebras can be met in applications together with C^*-algebras. By analogy with C^*-algebras locally C^*-algebras are applied to relativistic quantum mechanics [2, 4]. Therefore it seems useful to develop theory of Hilbert modules over other topological $*$-algebras as well. As we know the paper [15] of A. Mallios was the first one in this direction. In that paper finitely generated modules equipped with inner products over some topological $*$-algebras were considered in connection with Hermitian K-theory, the standard Hilbert module $l_2(A)$ over a locally C^*-algebra A was introduced and the index theory for elliptic operators over a locally C^*-algebra was constructed.

In the present paper the notion of a Hilbert module over a locally C^*-algebra is discussed and some results are obtained on this matter. In particular, it is proved that the set of adjointable endomorphisms of such modules is itself a locally C^*-algebra.

After the first version of this paper appeared we were informed that our main result was already known to specialists [22, 23, 27], but we think that our detailed proof may still be of interest.

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1 Locally C^*-algebras

We start with some information from [8, 1, 5] on locally C^*-algebras.

Definition 1.1. A complex algebra A is called a *LMC-algebra* if it is a separable locally convex space with respect to some family of seminorms $\{P_\alpha\}_{\alpha\in\Delta}$ satisfying the following condition:

(1) $P_\alpha(ab) \leq P_\alpha(a)P_\alpha(b)$ for all $a, b \in A$.

An involutive LMC-algebra A is called a *LMC $*$-algebra* if the following condition holds:

(2) $P_\alpha(a^*) = P_\alpha(a)$ for all $a \in A$ and $\alpha \in \Delta$.

A complete LMC $*$-algebra A is called a *locally C^*-algebra* (*LC^*-algebra*) if

(3) $P_\alpha(a^*a) = (P_\alpha(a))^2$ for all $a \in A$ and $\alpha \in \Delta$.

A *family of C^*-seminorms* is the family of seminorms $\{P_\alpha\}_{\alpha\in\Delta}$ satisfying condition (3) of Definition 1.1.

Let us give some examples of locally C^*-algebras [8, 1, 5].

Example 1.1. Any C^*-algebra is a LC^*-algebra.

Example 1.2. A closed $*$-subalgebra of a LC^*-algebra is a LC^*-algebra.

Example 1.3. Let M be a completely regular k-space ([12], p. 300) and $C(M)$ an algebra of all continuous complex-valued functions on M. For each compact space $K \subset M$ put

$$q_K(f) := \sup_{x\in K} |f(x)|, \quad f \in C(M).$$

Then the function q_K is a C^*-seminorm on $C(M)$ and with respect to the family of these seminorms $C(M)$ is a LC^*-algebra.

Example 1.4. Let Λ be a directed set of indices, $\{H_\lambda\}_{\lambda\in\Lambda}$ a family of Hilbert spaces such that $H_\lambda \subseteq H_\mu$ and

$$(\cdot,\cdot)_\lambda = (\cdot,\cdot)_\mu|_{H_\lambda}$$

for $\lambda \leq \mu$. Here $(\cdot,\cdot)_\lambda$ is an inner product on H_λ, $\lambda \in \Lambda$. Consider a locally convex space

$$H := \varinjlim{}_\lambda H_\lambda = \bigcup_\lambda H_\lambda.$$

The space H equipped with the topology of the inductive limit is called *a locally Hilbert space.* We denote by $L(H)$ the set of all linear continuous operators $T : H \to H$ such that

$$T = \varinjlim{}_\lambda T_\lambda, \quad T_\lambda \in B(H_\lambda).$$

Here $B(H_\lambda)$ is the space of all linear bounded operators on H_λ and $(T_\lambda)_{\lambda\in\Lambda}$ is the inductive family of the operators $T_\lambda \in B(H_\lambda)$. It is clear that $L(H)$ is an algebra. Furthemore, if $(T_\lambda)_{\lambda\in\Lambda}$ is an inductive family of linear bounded operators on $H_\lambda, \lambda \in \Lambda$, then the family of adjoint operators $(T^*_\lambda)_{\lambda\in\Lambda}$ is inductive too. The map

$$* : L(H) \to L(H), \quad T \mapsto T^* = \varinjlim{}_\lambda T^*_\lambda$$

defines an involution on $L(H)$. If $||\cdot||_\lambda$ is the operator norm on $B(H_\lambda)$ then the function

$$q_\lambda(T) := ||T_\lambda||_\lambda, \; T \in L(H)$$

is a C^*-seminorm on $L(H)$ for each $\lambda \in \Lambda$ and $L(H)$ is a LC^*-algebra with respect to the family of seminorms $\{q_\lambda\}_{\lambda\in\Lambda}$.

Theorem 1.1. ([8], Theorem 5.1). *Any LC^*-algebra is isomorphic to a closed $*$-subalgebra of $L(H)$ for some locally Hilbert space H.*

Let A be a LC^*-algebra with respect to a family of C^*-seminorms $\{P_\alpha\}_{\alpha\in\Delta}$. We denote by I_α the kernel of the seminorm P_α, i.e. the set of elements $a \in A$ such that $P_\alpha(a) = 0$. It is clear that I_α is a closed $*$-ideal in A. Therefore the quotient space $A_\alpha = A/I_\alpha$ is a normed $*$-algebra with respect to the norm

$$||a_\alpha|| := P_\alpha(a), \quad a_\alpha = a + I_\alpha \in A_\alpha.$$

It follows from Theorem 2.4 of [1] that the algebra A_α is complete, i.e. it is a C^*-algebra.

By e we will denote the identity element of A. Clearly $P_\alpha(e) = 1$ for any non-zero seminorm P_α.

If A is an algebra without the identity element, then by A^+ we denote its unitalization. By Theorem 2.3 of [8], any seminorm P_α can be extended up to a C^*-seminorm P_α^+ on A^+ and A^+ is a locally C^*-algebra with respect to the family of seminorms P_α^+.

The *spectrum* of an element a of a unital LC^*-algebra A is the set $Sp(a) = Sp_A(a)$ of complex numbers z such that $a - z \cdot 1$ is not invertible. If A is a non-unital algebra, then the *spectrum* of an element $a \in A$ is its spectrum in the LC^*-algebra A^+. It follows from Corollary 2.1 of [8] that

$$Sp_A(a) = \bigcup_{\alpha\in\Delta} Sp_{A_\alpha}(a_\alpha), \quad a_\alpha = a + I_\alpha \tag{1.1}$$

for each $a \in A$. An element $a \in A$ is called *positive* (and is written $a \geq 0$) if it is Hermitian, i.e. $a = a^*$ and one of the following equivalent (see [8], Proposition 2.1) conditions is true:

(1) $Sp(a) \subset [0, \infty)$;

(2) $a = b^*b$ for some $b \in A$;

(3) $a = h^2$ for some Hermitian $h \in A$.

Besides, the set of positive elements $P^+(A)$ is a closed convex cone in A and $P^+(A) \bigcap (-P^+(A)) = \{0\}$.

Remark 1.1. If $a \in A$ is a positive element then there exists a unique positive element $h \in A$ satisfying the condition (3). This element is called a *square root* of a and is denoted by $a^{\frac{1}{2}} = \sqrt{a}$. For elements $a, b \in A$ the inequality $a \geq b$ (or $b \leq a$) means that $a - b \geq 0$.

Lemma 1.1. ([8]).

(a) If $a, b \in P^+(A)$ and $a \leq b$, then $P_\alpha(a) \leq P_\alpha(b)$ for all $\alpha \in \Delta$.

(b) If $e \in A, b \in A$ and $b \geq e$ then the element b is invertible and $b^{-1} \leq e$.

(c) If elements $a, b \in A$ are invertible and $0 \leq a \leq b$ then $a^{-1} \geq b^{-1}$.

(d) If $a, b, c \in A$ and $a \leq b$ then $c^*ac \leq c^*bc$.

Lemma 1.2. *Let A be a unital locally C^*-algebra, $a \in P^+(A)$ and t be a positive number. Then for any $\alpha \in \Delta$ the following relations hold:*

(a) $P_\alpha((e + ta)^{-1}) \leq 1$;

(b) $P_\alpha(a(e + a)^{-1}) \leq 1$;

(c) $P_\alpha(e - a) \leq 1$ if $P_\alpha(a) \leq 1$.

Proof. Since $e + ta \geq e$, we obtain from (b) and (a) of Lemma 1.1 that the element $e + ta$ is invertible and

$$P_\alpha((e+ta)^{-1}) \leq P_\alpha(e) = 1.$$

Using $e + a \geq a$ and statement (d) of Lemma 1.1 for $c = \sqrt{(e+a)^{-1}}$ we obtain

$$a(e+a)^{-1} \leq e.$$

This yields (b).

We will prove (c). Since $A_\alpha = A/I_\alpha$ is a C^*-algebra with the identity element $e_\alpha = e + I_\alpha$ and $||a_\alpha|| = P_\alpha(a) \leq 1$, we obtain that

$$P_\alpha(e-a) = ||e - a + I_\alpha|| = ||e_\alpha - a_\alpha|| \leq 1.$$

The lemma is proved.

We denote by A^s the set of all elements $a \in A$ such that

$$||a||^s := \sup_{\lambda \in \Delta} P_\alpha(a) < \infty.$$

Then A^s is a $*$-subalgebra of A and $|| \cdot ||^s$ is a norm for A^s. Moreover, the following theorem is true.

Theorem 1.2. ([1], Theorem 2.3). *The algebra A^s is dense in A and is a C^*-algebra with respect to the norm $|| \cdot ||^s$.*

An *approximate identity* of a locally C^*-algebra A is any increasing net $\{u_\lambda\}_{\lambda \in \Lambda}$ of positive elements such that

1) $P_\alpha(u_\lambda) \leq 1$ for all $\alpha \in \Delta, \lambda \in \Lambda$;

2) $\lim\limits_\lambda (a - au_\lambda) = \lim\limits_\lambda (a - u_\lambda a) = 0$ for any $a \in A$.

Any locally C^*-algebra (and its closed ideal) has an approximate identity (see [8], Theorem 2.6).

2 Hilbert modules over LC^*-algebras

Let A be a LC^*-algebra with respect to the family of C^*-seminorms $\{P_\alpha\}_{\alpha \in \triangle}$. Below we will assume that the algebra A has the unit e. Considering A^+ instead of A, one can easily extend all results for the case of non-unital algebras.

Definition 2.1. Let X be a right A-module. An *A-valued inner product* on X is a map $< .,. >: X \times X \to A$ satisfying the following conditions:

(1) $< x, x > \geq 0$ for all $x \in X$;

(2) $< x, x > = 0$ if and only if $x = 0$;

(3) $< x_1 + x_2, y > = < x_1, y > + < x_2, y >$ for all $x_1, x_2, y \in X$;

(4) $< xa, y > = < x, y > a$ for all $x, y \in X, a \in A$;

(5) $< x, y >^* = < y, x >$ for all $x, y \in X$.

A right A-module equipped with an A-valued inner product is called a *pre-Hilbert A-module.*

Lemma 2.1. *Let X be a pre-Hilbert A-module. Then for each $\alpha \in \triangle$ and for all $x, y \in X$ the following Cauchy-Bunyakovskii inequality holds:*

$$P_\alpha(< x, y >)^2 \leq P_\alpha(< x, x >) P_\alpha(< y, y >). \tag{2.1}$$

Proof. Suppose $x, y \in X$, $b \in A$. Let us consider the expression

$$< x + yb, x + yb >=< x, x > +b^* < x, y > + < y, x > b + b^* < y, y > b \geq 0. \tag{2.2}$$

Assuming $P_\alpha(< y, y >) \neq 0$, we put

$$b = -\frac{< x, y >}{P_\alpha(< y, y >)}$$

in (2.2). It now follows that

$$< x, x > -\frac{< y, x >< x, y >}{P_\alpha(< y, y >)} - \frac{< y, x >< x, y >}{P_\alpha(< y, y >)} + \frac{< y, x >< y, y >< x, y >}{P_\alpha(< y, y >)^2} \geq 0,$$

whence, using item (a) of Lemma 1.1, we obtain

$$P_\alpha\left(\frac{2 < y, x >< x, y >}{P_\alpha(< y, y >)}\right) \leq P_\alpha\left(< x, x > + \frac{< y, x >< y, y >< x, y >}{P_\alpha(< y, y >)^2}\right).$$

Therefore,

$$\frac{2P_\alpha(< x, y >)^2}{P_\alpha(< y, y >)} \leq P_\alpha(< x, x >) + \frac{P_\alpha(< y, x >) P_\alpha(< y, y >) P_\alpha(< x, y >)}{P_\alpha(< y, y >)^2}$$

$$= P_\alpha(< x, x >) + \frac{P_\alpha(< x, y >)^2}{P_\alpha(< y, y >)}.$$

This implies inequality (2.1). If $P_\alpha(< x, x >) \neq 0$, then it is true by the same reason.

Let now $P_\alpha(< x, x >) = P_\alpha(< y, y >) = 0$. Putting $b = - < x, y >$ in (2.2), we get

$$< x, x > + < y, x >< y, y >< x, y >\geq 2 < y, x >< x, y >,$$

whence

$$2P_\alpha(< x, y >)^2 = 2P_\alpha(< y, x >< x, y >) \leq P_\alpha(< x, x >)+$$
$$+P_\alpha(< y, x >) P_\alpha(< y, y >) P_\alpha(< x, y >) = 0.$$

Thus $P_\alpha(< x, y >) = 0$ and inequality (2.1) is true in this case too. The lemma is proved.

Lemma 2.2. *Let X be a pre-Hilbert A-module with respect to an inner product $< ., . >$. Put*

$$\bar{P}_\alpha(x) := P_\alpha(< x, x >)^{\frac{1}{2}} \tag{2.3}$$

for any $\alpha \in \Delta$. Then the function $\bar{P}_\alpha$ is a seminorm on X and the following conditions hold:

(1) $\bar{P}_\alpha(xa) \leq \bar{P}_\alpha(x)P_\alpha(a)$ *for all* $x \in X,\ a \in A$;
(2) if $\bar{P}_\alpha(x) = 0$ *for all* $\alpha \in \Delta$, *then* $x = 0$;
(3) $\bar{P}_\alpha(x) = \sup_{\bar{P}_\alpha(y)\leq 1} P_\alpha(< x, y >)$ *for all* $x \in X,\ \alpha \in \Delta$.

Proof. Under the axioms of seminorm we have

$$\bar{P}_\alpha(x) = P_\alpha(< x, x >)^{\frac{1}{2}} \geq 0$$

and

$$\bar{P}_\alpha(\lambda x) = \sqrt{P_\alpha(< \lambda x, \lambda x >)} = \sqrt{P_\alpha(\bar{\lambda} < x, x > \lambda)}$$
$$= \sqrt{P_\alpha(|\lambda|^2 < x, x >)} = \sqrt{|\lambda|^2 P_\alpha(< x, x >)} = |\lambda|\bar{P}_\alpha(x)$$

for all $x \in X$ and $\lambda \in \mathbf{C}$.

Using the Cauchy-Bunyakovskii inequality one has

$$\bar{P}_\alpha(x + y) = \sqrt{P_\alpha(< x + y, x + y >)}$$
$$= \sqrt{P_\alpha(< x, x > + < y, x > + < x, y > + < y, y >)}$$
$$\leq \sqrt{P_\alpha(< x, x >) + 2P_\alpha(< x, y >) + P_\alpha(< y, y >)}$$
$$\leq \sqrt{\bar{P}_\alpha(x)^2 + 2\bar{P}_\alpha(x)\bar{P}_\alpha(y) + \bar{P}_\alpha(y)^2} = \bar{P}_\alpha(x) + \bar{P}_\alpha(y)$$

for all $x, y \in X$. Thus $\bar{P}_\alpha$ is a seminorm on X.

Let us prove (1). For all $a \in A$ and $x \in X$ we derive

$$\bar{P}_\alpha(xa) = \sqrt{P_\alpha(< xa, xa >)} = \sqrt{P_\alpha(a^* < x, x > a)} \leq \sqrt{P_\alpha(a^*)P_\alpha(< x, x >)P_\alpha(a)} = P_\alpha(a)\bar{P}_\alpha(x).$$

Suppose $x \in X$ and $\bar{P}_\alpha(x) = 0$ for all $\alpha \in \triangle$. Then $P_\alpha(< x, x >) = 0$ for all $\alpha \in \triangle$. Therefore $< x, x >= 0$ and, consequently, $x = 0$. Thus (2) is true. Equality (3) follows easy from the Cauchy-Bunyakovskii inequality. The lemma is proved.

Lemma 2.2 implies that X is a separable locally convex space with respect to the family of seminorms $\{\bar{P}_\alpha : \alpha \in \triangle\}$.

Definition 2.2. Let X be a pre-Hilbert A-module equipped with the inner product $< .,. >$. If X is a complete locally convex space with respect to the family of seminorms $\{\bar{P}_\alpha\}_{\alpha\in\triangle}$ defined by (2.3), then it is called a *Hilbert A-module.*

Example 2.1. Any closed right ideal I of a locally C^*-algebra A equipped with the inner product $< a, b >= a^*b$ is a Hilbert A-module.

Example 2.2 ([15]). Let $l_2(A)$ be the set of all sequences $x = (x_n)_{n\in\mathbf{N}}$ of elements from a locally C^*-algebra A such that the series

$$\sum_{i=1}^{\infty} x_i^* x_i$$

is convergent in A. Then $l_2(A)$ is a right Hilbert A-module with respect to the pointwise operations and the inner product

$$< x, y >= \sum_{i=1}^{\infty} x_i^* y_i.$$

Let A be a locally C^*-algebra and X a right Hilbert A-module equipped with the inner product $< .,. >$. We will denote by X^s the set of all $x \in X$ such that $< x, x > \in A^s$. It is verified immediately that X^s is A^s-module.

Lemma 2.3. *For all $x \in X$ and $\alpha \in \Delta$ one has:*

a) $\bar{P}_\alpha(x(e + \sqrt{< x, x >})^{-1}) \le 1$;

b) $\lim_{t \to +0} \bar{P}_\alpha(x - x(e + t\sqrt{< x, x >})^{-1}) = 0.$

Proof. a) Let $x \in X$ be an arbitrary element. Then for each $\alpha \in \Delta$ we have by statement (b) of Lemma 1.2 that

$$\bar{P}_\alpha(x(e + \sqrt{< x, x >})^{-1})^2 = P_\alpha(< x(e + \sqrt{< x, x >})^{-1}, x(e + \sqrt{< x, x >})^{-1} >)$$

$$= P_\alpha((e + \sqrt{< x, x >})^{-1} < x, x > (e + \sqrt{< x, x >})^{-1})$$

$$= P_\alpha(\sqrt{< x, x >}(e + \sqrt{< x, x >})^{-1})^2 \le 1.$$

b) For each $x \in X$ and each positive number t we have

$$x - x(e + t\sqrt{< x, x >})^{-1} = (x(e + t\sqrt{< x, x >}) - x)(e + t\sqrt{< x, x >})^{-1}$$

$$= tx\sqrt{< x, x >}(e + t\sqrt{< x, x >})^{-1}.$$

Therefore by statement (a) of Lemma 1.2

$$\bar{P}_\alpha(x - x(e + t\sqrt{< x, x >})^{-1}) \le t\bar{P}_\alpha(x\sqrt{< x, x >}).$$

This implies b). The lemma is proved.

Corollary 2.1. *The set X^s is dense in X.*

Proof. Let $x \in X$ be an arbitrary element and t a positive number. Then Lemma 2.3 implies that the elements $(e + \sqrt{< x, x >})^{-1}$, $x(e + t\sqrt{< x, x >})^{-1}$ belong to X^s and

$$\lim_{t \to 0}(x(e + t\sqrt{< x, x >})^{-1}) = x$$

in X. Thus X^s is dense in X.

Theorem 2.1. *X^s is a Hilbert C^*-module over the C^*-algebra A^s.*

Proof. First we will show that the restriction of the inner product $< .,. >$ from X to X^s is an A^s-valued inner product on X^s. Indeed, by the Cauchy-Bunyakovskii inequality (2.1) we have

$$P_\alpha(< x, y >)^2 \le P_\alpha(< x, x >) \cdot P_\alpha(< y, y >) \le || < x, x > ||^s || < y, y > ||^s$$

for all $x, y \in X^s$ and $\alpha \in \Delta$. Therefore, $< x, y > \in A^s$.

Let us prove completeness of X^s with respect to the norm

$$||x||^s = (|| < x, x >)||^s)^{\frac{1}{2}} = \sup_{\alpha \in \Delta} \bar{P}_\alpha(x). \tag{2.4}$$

Let $\{x_n\}$ be a fundamental sequence in X^s, i.e. for any $\varepsilon > 0$ there exists a natural number n_ε such that $||x_m - x_n||^s < \varepsilon$ if $m, n > n_\varepsilon$, whence

$$\bar{P}_\alpha(x_m - x_n) \le ||x_m - x_n||^s < \varepsilon$$

for all $\alpha \in \triangle$ and $m, n > n_\varepsilon$. This means that $\{x_n\}$ is a Cauchy sequence in X and as X is complete, so we conclude that the limit

$$x = \lim_{n\to\infty} x_n$$

exists in X. It follows from the inequalities

$$\left|\bar{P}_\alpha(x_m) - \bar{P}_\alpha(x_n)\right| \leq \bar{P}_\alpha(x_m - x_n), \ | \ ||x_m||^s - ||x_n||^s \ | \leq ||x_m - x_n||^s$$

that the sequences $\{\bar{P}_\alpha(x_n)\}$ and $\{||x_n||^s\}$ are Cauchy sequences of numbers and therefore are convergent. Besides, for each $\alpha \in \triangle$

$$\bar{P}_\alpha(x) = \lim_{n\to\infty} \bar{P}_\alpha(x_n) \leq \lim_{n\to\infty} ||x_n||^s < \infty$$

hence $x \in X^s$. For all $\alpha \in \triangle$ and $n > n_\varepsilon$ we have

$$\bar{P}_\alpha(x - x_n) = \lim_{m\to\infty} \bar{P}_\alpha(x_m - x_n) \leq \varepsilon.$$

Thus for $n > n_\varepsilon$

$$||x - x_n||^s = \sup_{\alpha\in\triangle} \bar{P}_\alpha(x - x_n) \leq \varepsilon.$$

This means that the sequence $\{x_n\}$ converges to the element x with respect to the topology of X^s. The theorem is proved.

For any $\alpha \in \triangle$ let us put

$$I_\alpha = \{a \in A : \ P_\alpha(a) = 0\}, \qquad J_\alpha = I_\alpha \cap A^s,$$

$$\bar{I}_\alpha = \{x \in X : < x, x > \in I_\alpha\}, \qquad \bar{J}_\alpha = \bar{I}_\alpha \cap X^s.$$

The subset $\bar{I}_\alpha$ is a closed A-submodule in X, J_α is a closed ideal of the C^*-algebra A^s and $\bar{J}_\alpha$ is a closed A^s-submodule in X^s. Therefore the quotient $X_\alpha := X/\bar{I}_\alpha$ is is a normed space with respect to the norm

$$||x + \bar{I}_\alpha|| := \inf_{y\in\bar{I}_\alpha} \bar{P}_\alpha(x + y) = \bar{P}_\alpha(x), \qquad x \in X,$$

and the quotient $X^s_\alpha := X^s/\bar{J}_\alpha$ is a Banach space with respect to the norm

$$||x + \bar{J}_\alpha|| = \inf_{y\in\bar{J}_\alpha} ||x + y||^s, \qquad x \in X^s.$$

Lemma 2.4. *Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate identity in the ideal J_α of the C^*-algebra A^s. Then*

a) $\lim_\lambda ||y - yu_\lambda||^s = 0$ *for any* $y \in \bar{J}_\alpha$.

b) $||x + \bar{J}_\alpha|| = \lim_\lambda ||x - xu_\lambda||^s$ *for any* $x \in X^s$.

c) $||a + J_\alpha|| = \lim_\lambda ||a - au_\lambda||^s$ *for any* $a \in A^s$.

Proof. a) Suppose $y \in \bar{J}_\alpha$. Then for any $\beta \in \Delta$ we have

$$\bar{P}_\beta(y - yu_\lambda)^2 = P_\beta(< y - yu_\lambda,\ y - yu_\lambda >) = P_\beta((e - u_\lambda) < y, y > (e - u_\lambda))$$

$$\leq P_\beta(< y, y > - < y, y > u_\lambda) \leq || < y, y > - < y, y > u_\lambda||^s$$

because $P_\beta(e - u_\lambda) \leq ||e - u_\lambda||^s \leq 1$. Since β is arbitrary, we get

$$(||y - yu_\lambda||^s)^2 \leq || < y, y > - < y, y > u_\lambda||^s.$$

Using $< y, y > \in J_\alpha$ we obtain $\lim\limits_\lambda ||y - yu_\lambda||^s = 0$.

b) Let now $x \in X^s$ and take $\varepsilon > 0$. By definition of infimum there exists an element $y \in \bar{J}_\alpha$ such that

$$||x + y||^s < ||x + \bar{J}_\alpha|| + \frac{\varepsilon}{2}.$$

Item a) implies that there exists $\lambda_0 \in \Lambda$ such that for $\lambda > \lambda_0$ one has

$$||y - yu_\lambda||^s < \frac{\varepsilon}{2}.$$

Then for all $\lambda > \lambda_0$

$$||x - xu_\lambda||^s = ||x(e - u_\lambda) + y(e - u_\lambda) - y(e - u_\lambda)||^s \leq ||(x + y)(e - u_\lambda)||^s + ||y - yu_\lambda||^s$$

$$\leq ||(x + y)||^s + ||y - yu_\lambda||^s \leq ||x + \bar{J}_\alpha|| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = ||x + \bar{J}_\alpha|| + \varepsilon.$$

Therefore for $\lambda > \lambda_0$ we have

$$\Big| ||x + \bar{J}_\alpha|| - ||x - xu_\lambda||^s \Big| < \varepsilon.$$

Thus statement b) is true. Statement c) follows from b) for $X = A$. The lemma is proved.

Lemma 2.5. *For any $x \in X^s$ the equality*

$$||x + \bar{J}_\alpha|| = \bar{P}_\alpha(x)$$

holds.

Proof. Let $x \in X^s$ and $\{u_\lambda\}_{\lambda\in\Lambda}$ approximate identity of the ideal J_α. Then by statements b) and c) of Lemma 2.4 we have

$$||x + \bar{J}_\alpha||^2 = \lim_\lambda(||x - xu_\lambda||^s)^2 = \lim_\lambda \sup_\beta \bar{P}_\beta(x - xu_\lambda)^2 = \lim_\lambda \sup_\beta P_\beta(< x - xu_\lambda, x - xu_\lambda >)$$

$$= \lim_\lambda \sup_\beta P_\beta((e - u_\lambda) < x, x > (e - u_\lambda)) \leq \lim_\lambda \sup_\beta P_\beta(< x, x > - < x, x > u_\lambda)$$

$$= \lim_\lambda || < x, x > - < x, x > u_\lambda||^s = || < x, x > + J_\alpha|| = P_\alpha(< x, x >) = (\bar{P}_\alpha(x))^2.$$

The next to last equality follows from uniqueness of C^*-norm of C^*-algebra A^s/J_α (see [1]). Thus we have proved that

$$||x + \bar{J}_\alpha|| \leq \bar{P}_\alpha(x).$$

The inverse inequality is verified immediately:

$$\bar{P}_\alpha(x) = ||x + \bar{I}_\alpha|| = \inf_{y \in \bar{I}_\alpha} \bar{P}_\alpha(x+y) \le \inf_{y \in \bar{J}_\alpha} \bar{P}_\alpha(x+y) \le \inf_{y \in \bar{J}_\alpha} ||x+y||^s = ||x + \bar{J}_\alpha||.$$

The lemma is proved.

Theorem 2.2. *The quotient module $X_\alpha = X/\bar{I}_\alpha$ is a Hilbert module over the C^*-algebra A_α.*

Proof. We define an action of the algebra A_α on X_α by the formula

$$(x + \bar{I}_\alpha)(a + I_\alpha) := xa + \bar{I}_\alpha, \quad x \in X, \quad a \in A.$$

With respect to this action X_α is a right A_α-module. Let us define the A_α-valued inner product in X_α by the following formula

$$< x + \bar{I}_\alpha,\ y + \bar{I}_\alpha >:=< x, y > + I_\alpha, \quad x, y \in X.$$

The inner product axioms are easily verified. Positive definiteness follows from equality (1.1). Further,

$$|| < x + \bar{I}_\alpha,\ x + \bar{I}_\alpha > || = || < x, x > + I_\alpha|| = P_\alpha(< x, x >) = (\bar{P}_\alpha(x))^2 = ||x + \bar{I}_\alpha||^2,$$

i.e.

$$||x + \bar{I}_\alpha|| = || < x + \bar{I}_\alpha,\ x + \bar{I}_\alpha > ||^{\frac{1}{2}}.$$

To establish the completeness of X_α let us consider the map

$$\varphi : X^s/\bar{J}_\alpha \to X/\bar{I}_\alpha = X_\alpha$$

defined by the formula

$$\varphi(x + \bar{J}_\alpha) = x + \bar{I}_\alpha, \quad x \in X^s.$$

It is clear that φ is an injective linear map. Besides, it follows from Lemma 2.5 that this map preserves the norm,

$$||\varphi(x + \bar{J}_\alpha)|| = ||x + \bar{I}_\alpha|| = \bar{P}_\alpha(x) = ||x + \bar{J}_\alpha||, \quad x \in X^s.$$

Consequently, the image $\varphi(X^s/\bar{J}_\alpha) = \tilde{X}^s$ is closed in X_α. Let us prove that the set $\tilde{X}^s$ is dense in X_α.

Let $x \in X$ be an arbitrary element. Then Lemma 2.3 implies that for any $t > 0$ the element $x(e + t\sqrt{< x, x >})^{-1}$ belongs to X^s and

$$\lim_{t \to 0} x(e + t\sqrt{< x, x >})^{-1} = x$$

in X. Therefore,

$$x(e + t\sqrt{< x, x >})^{-1} + \bar{I}_\alpha \in \tilde{X}^s$$

and

$$\lim_{t \to 0} [x(e + t\sqrt{< x, x >})^{-1} + \bar{I}_\alpha] = x + \bar{I}_\alpha$$

in X_α. Since $\tilde{X}^s$ is closed in X_α, we conclude that $\tilde{X}^s = X_\alpha$. Thus the space X_α is complete. The theorem is proved.

3 Operators on Hilbert modules over locally C^*-algebras

Let X be a Hilbert module over a locally C^*-algebra A and $\{P_\alpha\}_{\alpha\in\Delta}$ a family of C^*-seminorms on A.

Definition 3.1. $\mathbf{C}$-linear A-homomorphism $T : X \to X$ is called *bounded operator* on the module X if for each $\alpha \in \Delta$ there exists a constant $C_\alpha > 0$ such that

$$\bar{P}_\alpha(Tx) \le C_\alpha \bar{P}_\alpha(x)$$

for all $x \in X$.

We denote by $\mathrm{End}_A(X)$ the set of all bounded operators on X. The following lemma is easily proved.

Lemma 3.1. *For any $\alpha \in \Delta$ the function*

$$\hat{P}_\alpha(T) := \sup_{\bar{P}_\alpha(x)\le 1} \bar{P}_\alpha(Tx)$$

is a seminorm in $\mathrm{End}_A(X)$.

Theorem 3.1. *The set* $\mathrm{End}_A(X)$ *is a complte LMC-algebra with respect to the family of seminorms* $\{\hat{P}_\alpha\}_{\alpha\in\Delta}$.

Proof. Let $T_1, T_2 \in \mathrm{End}_A(X)$. Then for all $\alpha \in \Delta$ and $x \in X$

$$\bar{P}_\alpha((T_1+T_2)x) \le \bar{P}_\alpha(T_1x) + \bar{P}_\alpha(T_2x) \le (\hat{P}_\alpha(T_1) + \hat{P}_\alpha(T_2))\bar{P}_\alpha(x),$$

$$\hat{P}_\alpha((T_1T_2)x) \le \hat{P}_\alpha(T_1)\bar{P}_\alpha(T_2x) \le \hat{P}_\alpha(T_1)\hat{P}_\alpha(T_2)\bar{P}_\alpha(x).$$

This implies that $T_1 + T_2,\ T_1T_2 \in \mathrm{End}_A(X)$ and

$$\hat{P}_\alpha(T_1+T_2) \le \hat{P}_\alpha(T_1) + \hat{P}_\alpha(T_2), \qquad \hat{P}_\alpha(T_1T_2) \le \hat{P}_\alpha(T_1)\hat{P}_\alpha(T_2). \tag{3.1}$$

Thus $\mathrm{End}_A(X)$ is an algebra and the function $\hat{P}_\alpha$ is a semi-multiplicative seminorm for this algebra.

Let us verify now that if $\hat{P}_\alpha(T) = 0$ for all $\alpha \in \Delta$, then $T = 0$.

Suppose $\hat{P}_\alpha(T) = 0$ for all $\alpha \in \Delta$. Then for all $x \in X$ and $\alpha \in \Delta$ we have

$$\bar{P}_\alpha(Tx) \le \hat{P}_\alpha(T)\bar{P}_\alpha(x) = 0,$$

cosequently, for all $x \in X,\ \alpha \in \Delta$

$$\bar{P}_\alpha(Tx) = 0,$$

hence $Tx = 0$ for all $x \in X$. This yields $T = 0$.

Let us prove completeness of $\mathrm{End}_A(X)$. Let $\{T_i\}_{i\in I} \subset \mathrm{End}_A(X)$ be a Cauchy net, i.e. for any $\alpha \in \Delta$ and $\varepsilon > 0$ there exists $\beta \in I$ such that for $i_1, i_2 > \beta$

$$\hat{P}_\alpha(T_{i_1} - T_{i_2}) < \varepsilon. \tag{3.2}$$

Therefore, if $\bar{P}_\alpha(x) \ne 0$ then for each $\varepsilon > 0$ there exists $\beta \in I$ such that for $i_1, i_2 > \beta$

$$\hat{P}_\alpha(T_{i_1} - T_{i_2}) < \frac{\varepsilon}{\bar{P}_\alpha(x)}.$$

Then for $i_1, i_2 > \beta$

$$\bar{P}_\alpha(T_{i_1}x - T_{i_2}x) \leq \bar{P}_\alpha(T_{i_1} - T_{i_2})\bar{P}_\alpha(x) < \varepsilon. \tag{3.3}$$

If $\bar{P}_\alpha(x) = 0$ then

$$\hat{P}_\alpha(T_{i_1}x - T_{i_2}x) \leq \hat{P}_\alpha(T_{i_1} - T_{i_2})\bar{P}_\alpha(x) = 0,$$

i.e. inequality (3.3) holds too.

This yields that for each $x \in X$ $\{T_i x\}_{i \in I} \subset X$ is a Cauchy net. Therefore there exists the limit

$$Tx = \lim_i T_i x. \tag{3.4}$$

It is easy to verify that the map T defined by (3.4) is an A-linear homomorphism on X. Let us prove boundedness of T.

Since for each $\alpha \in \Delta$ and $i_1, i_2 \in I$

$$|\hat{P}_\alpha(T_{i_1}) - \hat{P}_\alpha(T_{i_2})| \leq \hat{P}_\alpha(T_{i_1} - T_{i_2}),$$

we see that the net $\{\hat{P}_\alpha(T_i)\}_{i \in I}$ of positive numbers is a Cauchy net. Hence for each $\alpha \in \Delta$ the limit

$$\lim_i \hat{P}_\alpha(T_i) = \lambda_\alpha \geq 0$$

exists. For all $x \in X,\ \alpha \in \Delta$ we have

$$\bar{P}_\alpha(Tx) = \lim_i \bar{P}_\alpha(T_i x) \leq \lim_i (\hat{P}_\alpha(T_i)\bar{P}_\alpha(x)) = \lambda_\alpha \bar{P}_\alpha(x).$$

This yields boundedness of T. Thus $T \in \mathrm{End}_A(X)$.

It follows from (3.2) that for $P_\alpha(x) \leq 1$ and $i_1, i_2 > \beta$

$$\bar{P}_\alpha(T_{i_1}x - T_{i_2}x) \leq \sup_{\bar{P}_\alpha(y) \leq 1} \bar{P}_\alpha((T_{i_1} - T_{i_2})y) = \hat{P}_\alpha(T_{i_1} - T_{i_2}) < \varepsilon.$$

Taking limit over i_2 we obtain

$$\bar{P}_\alpha(T_{i_1}x - Tx) < \varepsilon$$

if $\bar{P}_\alpha(x) \leq 1$ and $i_1 > \beta$. Consequently, for $i_1 > \beta$ we have

$$\hat{P}_\alpha(T_{i_1} - T) = \sup_{\bar{P}_\alpha(x) \leq 1} \bar{P}_\alpha(T_{i_1}x - Tx) \leq \varepsilon,$$

whence

$$\lim T_i = T,$$

i.e. $\mathrm{End}_A(X)$ is a complete space. The theorem is proved.

Definition 3.2. An operator $T \in \mathrm{End}_A(X)$ is called *adjointable* if there exists an operator $T^* \in \mathrm{End}_A(X)$ such that

$$< Tx, y > = < x, T^* y >$$

for all $x, y \in X$.

We will denote by $\mathrm{End}^*_A(X)$ the set of all bounded adjointable operators on X.

Lemma 3.2. *Let $T : X \to X,\ T^* : X \to X$ be maps such that the equality*

$$< x, Ty > = < T^* x, y >$$

holds for any elements x and y from X. Then T and T^* are bounded operators on X, consequently, $T \in \mathrm{End}^*_A(X)$.

Proof. For all $x, y, z \in X,\ \lambda \in \mathbf{C}$ and $a \in A$ the following equalities hold

$$< z, T(x + y) > = < T^* z, x + y > = < T^* z, x > + < T^* z, y >$$

$$= < z, Tx > + < z, Ty > = < z, Tx + Ty >,$$

$$< z, T(\lambda x) > = < T^* z, \lambda x > = < T^* z, x > \lambda = < z, Tx > \lambda = < z, \lambda Tx >,$$

$$< z, T(xa) > = < T^* z, xa > = < T^* z, x > a = < z, Tx > a = < z, (Tx)a > .$$

As z was arbitrary, so we obtain

$$T(x + y) = Tx + Ty, \quad T(\lambda x) = \lambda Tx,$$

$$T(xa) = (Tx)a.$$

Thus the desired properties of linearity hold.

Now let us prove that the submodule $\bar{I}_\alpha \subset X$ is invariant with respect to T and T^* for any $\alpha \in \Delta$. Indeed, if $x \in \bar{I}_\alpha$, i.e. $\bar{P}_\alpha(x) = 0$, then by the Cauchy-Bunyakovskii inequality we have

$$\bar{P}_\alpha(Tx)^2 = P_\alpha(< Tx, Tx >) = P_\alpha(< x, T^* Tx >)$$

$$\leq P_\alpha(< x, x >)^{\frac{1}{2}} P_\alpha(< T^* Tx, T^* Tx >)^{\frac{1}{2}} = \bar{P}_\alpha(x) \bar{P}_\alpha(T^* Tx) = 0.$$

Consequently, $\bar{P}_\alpha(Tx) = 0$, i.e., $Tx \in \bar{I}_\alpha$. By the same reason we get $T^* x \in \bar{I}_\alpha$.

Thus maps

$$T_\alpha : X_\alpha \to X_\alpha, \qquad T^*_\alpha : X_\alpha \to X_\alpha$$

given by the formulas

$$T_\alpha(x + \bar{P}_\alpha) = Tx + \bar{I}_\alpha, \quad T^*_\alpha(x + \bar{I}_\alpha) = T^* x + \bar{I}_\alpha$$

are well-defined. Furthemore,

$$< x + \bar{I}_\alpha, T_\alpha(y + \bar{I}_\alpha) > = < T^*_\alpha(x + \bar{I}_\alpha), y + \bar{I}_\alpha >$$

for all $x, y \in X$. Since X_α is a Hilbert module over the C^*-algebra A_α, we obtain that T_α is a bounded A_α-homomorphism on X_α (see [17, 13]), i.e. there exists a constant $C_\alpha > 0$ such that

$$||T_\alpha(x + \bar{I}_\alpha)|| \leq C_\alpha ||x + \bar{I}_\alpha|| \tag{3.5}$$

for all $x \in X$. It follows from (3.5) that

$$\bar{P}_\alpha(Tx) \leq C_\alpha \bar{P}_\alpha(x)$$

for all $x \in X$ since $\bar{P}_\alpha(Tx) = ||Tx + \bar{I}_\alpha|| = ||T_\alpha(x + \bar{I}_\alpha)||$ and $||x + \bar{I}_\alpha|| = \bar{P}_\alpha(x)$, whence, since $\alpha \in \Delta$ is arbitrary, it follows that the operator T is bounded. The lemma is proved.

Theorem 3.2. *The set* $\mathrm{End}_A^*(X)$ *is a locally* C^**-algebra with respect to the family of seminorms* $\{\hat{P}_\alpha\}_{\alpha\in\Delta}$.

Proof. It is easy to verify that if $T, Q \in \mathrm{End}_A^*(X)$ and $\lambda \in C$, then

$$(T+Q)^* = T^* + Q^*, \quad (\lambda T)^* = \bar{\lambda} T^*,$$

$$(TQ)^* = Q^* T^*, \quad (T^*)^* = T.$$

Thus $\mathrm{End}_A^*(X)$ is a $*$-algebra. By statement (3) of Lemma 2.2 for each $T \in \mathrm{End}_A^*(X)$ we have

$$\hat{P}_\alpha(T^*T) = \sup_{\bar{P}_\alpha(x)\le 1} \bar{P}_\alpha(T^*Tx) = \sup_{\bar{P}_\alpha(x)\le 1}\ \sup_{\bar{P}_\alpha(y)\le 1} P_\alpha(< T^*Tx, y >)$$

$$= \sup_{\bar{P}_\alpha(x)\le 1}\ \sup_{\bar{P}_\alpha(y)\le 1} P_\alpha(< Tx, Ty >) \ge \sup_{\bar{P}_\alpha(x)\le 1} P_\alpha(< Tx, Tx >) = \sup_{\bar{P}_\alpha(x)\le 1} \bar{P}_\alpha(Tx)^2 = \hat{P}_\alpha(T)^2,$$

i.e.

$$\hat{P}_\alpha(T^*T) \ge \hat{P}_\alpha(T)^2. \tag{3.6}$$

It follows from (3.1) and (3.6) that

$$\hat{P}_\alpha(T^*T) = \hat{P}_\alpha(T)^2,$$

whence one can easy obtain that

$$\hat{P}_\alpha(T^*) = \hat{P}_\alpha(T).$$

Thus $\hat{P}_\alpha$ is a C^*-seminorm on $\mathrm{End}_A^*(X)$ for any $\alpha \in \Delta$.

It remains to prove completeness of $\mathrm{End}_A^*(X)$. For this aim it is enough to verify that $\mathrm{End}_A^*(X)$ is closed in $\mathrm{End}_A(X)$. Let $\{T_i\}_{i\in I} \subset \mathrm{End}_A^*(X)$ and $\lim T_i = T$. Then $\{T_i^*\}_{i\in I}$ is a Cauchy net in $\mathrm{End}_A(X)$ and therefore converges to some operator $Q \in \mathrm{End}_A(X)$. For all $x, y \in X$ and $i \in I$ we have

$$< T_i x, y > = < x, T_i^* y >$$

therefore taking limit over i we obtain

$$< Tx, y > = < x, Qy > .$$

Hence the operator T is adjointable and $T^* = Q$. Consequently, $T \in \mathrm{End}_A^*(X)$. The theorem is proved.

For any $\alpha \in \Delta$ and $T \in \mathrm{End}_A^*(X)$ we define the map $T_\alpha : X_\alpha \to X_\alpha$ by the formula

$$T_\alpha(x + \bar{I}_\alpha) := Tx + \bar{I}_\alpha.$$

One can easy verify that $T_\alpha \in \mathrm{End}_{A_\alpha}^*(X_\alpha)$ (see the proof of Lemma 3.2) and the map

$$T \mapsto T_\alpha$$

is a $*$-homomorphism from the LC^*-algebra $\mathrm{End}_A^*(X)$ to the C^*-algebra $\mathrm{End}_{A_\alpha}^*(X_\alpha)$. Besides,

$$||T_\alpha|| = \hat{P}_\alpha(T).$$

Proposition 3.1. *Let $T \in \mathrm{End}_A^*(X)$. Then*

a) $Sp(T) = \bigcup_{\alpha\in\Delta} Sp(T_\alpha)$;

b) T is self-adjoint if and only if T_α is self-adjoint for any $\alpha \in \Delta$;

c) T is a positive element of the LC^*-algebra $\mathrm{End}_A^*(X)$ if and only if T_α is a positive element of the C^*-algebra $\mathrm{End}_{A_\alpha}^*(X_\alpha)$ for any $\alpha \in \Delta$.

Proposition 3.2. *For an operator $T : X \to X$ the following conditions are equivalent:*

1) T is a positive element of the LC^*-algebra $\mathrm{End}_A^*(X)$;

2) for any element $x \in X$ the inequality $< Tx, x > \geq 0$ holds, i.e. this element is positive in A.

One can prove Proposition 3.2 similarly to Lemma 4.1 of [11] or by using statement c) of Proposition 3.1.

Proposition 3.3. *The map $T \mapsto T|_{X^s}$ is an isomorphism of the C^*-algebras $(\mathrm{End}_A^*(X))^s$ and $\mathrm{End}_{A^s}^*(X^s)$.*

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